

Corrections to *ghm22@cam.ac.uk*. Star (★) indicates a harder question.

- 1 Use the Cauchy–Schwarz inequality and the properties of the inner product to prove the triangle inequality

$$|x + y| \leq |x| + |y|$$

for a complex vector space, where $|x|$ is the norm of the vector x . Under what conditions does equality hold?

- 2 Given a set of vectors $u_1, u_2, \dots, u_m$ ($m \geq n$) that span an n -dimensional vector space, show that an orthogonal basis may be constructed by the Gram-Schmidt procedure:

$$e_1 = u_1$$

$$e_r = u_r - \sum_{s=1}^{r-1} \frac{e_s \cdot u_r}{e_s \cdot e_s} e_s \quad \text{for } r > 1.$$

What is the interpretation if any of the vectors e_r vanishes?

Find an orthonormal basis for the subspace of a four-dimensional Euclidean space spanned by the three vectors with components $(1, 1, 0, 0)$, $(0, 1, 2, 0)$ and $(0, 0, 3, 4)$.

- 3 An $n \times n$ complex matrix A is such that each row and each column has exactly one non-zero element. The Hermitian conjugate of A is $A^\dagger = (A^T)^*$ (where A^T is the transpose of A , and A^* is its complex conjugate). Show that $A^\dagger A$ is a real diagonal matrix.
- 4★ What does it mean to say that the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are linearly independent?

Let A be a linear operator on an n -dimensional vector space, having n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding eigenvectors $e_1, e_2, \dots, e_n$. Consider the action of the operator $A - \lambda_i I$ (where I is the identity operator) on the vector e_j in the cases $i = j$ and $i \neq j$.

Describe the action of this operator on $\sum_j \alpha_j \mathbf{e}_j$. Hence or otherwise show that the vectors $\{\mathbf{e}_i\}$ are linearly independent.

- 5 An $n \times n$ Hermitian matrix A is said to be positive definite if $\mathbf{x}^\dagger A \mathbf{x} > 0$ for all non-zero vectors $\mathbf{x} \in \mathbb{C}^n$.

Show that if A is positive definite, then any principal submatrix (a square matrix formed by deleting the same set of rows and columns from A) is also positive definite. Deduce that all diagonal elements A_{ii} must be strictly positive real numbers.

For a 2×2 positive definite matrix A , prove that $\det A \leq A_{11}A_{22}$. Under what condition does equality hold?

- 6 An Hermitian matrix A is one for which $A^\dagger = A$. Suppose that A and B are both Hermitian matrices. Show that $AB + BA$ is Hermitian. Also show that AB is Hermitian if and only if A and B commute.
- 7 Suppose A is an $n \times n$ matrix such that $A^2 = A$. By considering the eigenvalues of A show that either:
- $\det A = 1$ and $\text{Tr } A = n$; or,
 - $\det A = 0$ and $\text{Tr } A = m$ with $m < n$.

Show that the matrix $B = I - A$ also satisfies $B^2 = B$. Hence, show that any vector $\mathbf{x}$ can be uniquely decomposed as $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $A\mathbf{y} = \mathbf{y}$ and $A\mathbf{z} = \mathbf{0}$. What are the dimensions of the subspaces containing $\mathbf{y}$ and $\mathbf{z}$ in case (b)? Provide a geometric interpretation.

- 8 Show that two diagonalizable matrices A and B share a common basis of eigenvectors if and only if they commute. Use this result to deduce that the matrices $(I + A)$ and $(I - A)^{-1}$ commute.

- 9 An $n \times n$ complex matrix N is said to be nilpotent if $N^k = 0$ for some positive integer k .

By considering the eigenvalue equation $N\mathbf{x} = \lambda\mathbf{x}$ for a non-zero eigenvector $\mathbf{x}$, show that the only eigenvalue of a nilpotent matrix is $\lambda = 0$. Hence, determine $\det N$ and $\text{Tr } N$.

Show that the matrix $(I - N)$ is always invertible, and find an explicit expression for $(I - N)^{-1}$ as a finite sum involving powers of N .

- 10 In quantum mechanics, the statistical state of a system is often represented by a *density matrix* ρ . A valid density matrix must be Hermitian, positive semi-definite ($\mathbf{x}^\dagger \rho \mathbf{x} \geq 0$ for all $\mathbf{x}$), and satisfy $\text{Tr } \rho = 1$.

Show that the eigenvalues λ_i of ρ must satisfy $0 \leq \lambda_i \leq 1$.

By expressing the trace of ρ^2 in terms of these eigenvalues, prove the inequality:

$$\text{Tr } \rho^2 \leq 1$$

Under what condition regarding the eigenvalues does $\text{Tr } \rho^2 = 1$? (A state satisfying this condition is known as a *pure state*, whereas if $\text{Tr } \rho^2 < 1$, it is a *mixed state*. You'll see this topic in Part II Physics, if you do it.)

- 11 Let H be an $n \times n$ Hermitian matrix representing the Hamiltonian (energy operator) of a quantum system. The time evolution of the state vector $\psi(t) = U(t)\psi(0)$ is governed by the matrix exponential $U(t) = e^{-iHt}$.

Using the power series definition of the matrix exponential,

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!},$$

prove that $U(t)$ is a unitary matrix.

If the initial state vector $\psi(0)$ is normalized such that $|\psi(0)|^2 = \psi(0)^\dagger \psi(0) = 1$, show that the state vector remains normalized at all times t .